UDC 534

ON A CASE OF RESONANCE FOR NONLINEAR SYSTEMS

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V. S. NUSTROV
(Sverdlovsk)
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We examine the case of resonance for systems close to nonlinear systems, admitting of a parametric periodic solution. Among the eigenvalues of the matrix of the system's linear part there are zero and pure imaginary ones. We have proved (under certain conditions) the absence of a periodic solution for the original system for which the generating solution is trivial.

1. Consider the system

$$dx/dt = Ax + X(x) + \mu F(t, x, \mu) \tag{1.1}$$

Here x is an $(l+n)$-dimensional vector, A is a constant matrix, X is an analytic function of x in a sufficiently small neighborhood of the origin (of order no less than second); the function F depends analytically on x and on a small positive parameter μ, F is continuous and 2π-periodic in t. Let the equation

$$|A - E\rho| = 0 \tag{1.2}$$

have ($l - 2m$) zero roots and $2m$ roots of the form

$$\pm N_j \lambda \sqrt{-1} \quad (j = 1, \ldots, m) \tag{1.3}$$

(where N_j are integers), and for $\mu = 0$ let system (1.1) admit of a periodic solution with period T, depending on an arbitrary constant vector α

$$\alpha = \operatorname{col}(\alpha_1, \ldots, \alpha_\xi) \tag{1.4}$$

$$T = 2\pi\lambda^{-1}(1 + h(\alpha)) \tag{1.5}$$

We accept that for all the chosen eigenvalues of matrix A the elementary divisors are simple and that among the numbers N_j at least one equals unity.

We consider the question of the existence of periodic solutions of Eqs. (1.1), $x(t, \mu)$, $x(t, 0) = 0$, under the conditions of principal resonance [1], when it is known that there are no solutions which can be expanded into series in integral powers of μ. The existence of a parametric periodic solution of Eqs. (1.1) for $\mu = 0$ with period (1.5) can be ensured, for example, by the presence of $\xi - 1$ first integrals [2]. Equations (1.1) are of a more general class of systems than those close to Liapunov systems [1, 3]. In a formulation other than that in [1] the latter systems have been treated in (*) and in [4]. In [5] in the analysis of Eqs. (1.1) it was assumed that λ in (1.3) is an integer.

In the present paper we investigate the case when λ is not an integer, but among the quantities $N_j\lambda$ there are ($m - r$) integers $q_1, \ldots, q_{m-r}$. (Among the numbers q_i there

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*) Kleimenov, A. F., Oscillations of Time-Lag Systems Close to Liapunov Systems. Candidate Dissertation, Sverdlovsk, 1969.

can be multiple ones; we exclude the case [5] when all the numbers q_i are multiples of one of them). Then system (1.1) can be written as

$$dv/dt = Bv + V(v, z) + \mu P(t, v, z, \mu) \tag{1.6}$$

$$dz/dt = Cz + Z(v, z) + \mu Q(t, v, z, \mu) \tag{1.7}$$

$$B = \operatorname{diag}(B_0, B_1, \ldots, B_{m-r}), \qquad B_i = \begin{Vmatrix} 0 & -q_i \\ q_i & 0 \end{Vmatrix} \tag{1.8}$$

$$(i = 1, \ldots, m-r)$$

Here v is an $(l-2r)$-dimensional vector, z is an $(n+2r)$-dimensional vector, the functions V, Z, P, Q are of the same type as X, F in (1.1), B_0 is the zero $((l-2m)\times(l-2m))$-matrix among the eigenvalues of matrix C there are no quantities of the form $\pm q\sqrt{-1}$ and q is an integer. In the case being considered $\xi \leqslant l-2r$ in (1.4) since at $\mu = 0$ we can investigate, instead of (1.6), (1.7), the following system

$$dv/dt = Bv + V(v, z(v))$$

where $z = z(v)$ is a solution of the equation

$$(\partial z/\partial v, \ (Bv + V)) = Cz + Z$$

In what follows we use the following notation:

a) $\varphi(t)$ and $\psi(t)$ are the matrices of periodic solutions of the system

$$dv/dt = Bv$$

and the matrix adjoint to it;

b) p_* is the integer closest to λ $(p_* \neq 0)$;

c) $v = \operatorname{col}(v^{(1)}, v^{(2)}), \qquad v^{(1)} = \operatorname{col}(v_1, \ldots, v_{l-2m})$
$v^{(2)} = \operatorname{col}(v_{l-2m+1}, \ldots, v_{l-2r})$

(the same notation is used below for some other functions);

d) $[f(t)] = f(2\pi) - f(0)$

We assume that the condition of principal resonance [1, 5]

$$\delta = \operatorname{col}(\delta_1, \ldots, \delta_{l-2r}) = \frac{1}{2\pi}\int_0^{2\pi} \psi'(t) P(t, 0, 0, 0)\, dt \neq 0 \tag{1.9}$$

is satisfied.

2. Theorem 2.1. If among the numbers $q_1, \ldots, q_{m-r}$ there are no multiples of the quantity $\lambda(\lambda - p_*)^{-1}$ and the conditions

$$V^{(1)}(v, 0) = 0, \qquad \delta^{(1)} \neq 0 \tag{2.1}$$

are satisfied, then system (1.6) does not have periodic solutions $x(t, \mu)$, $x(t, 0) = 0$.

For example, we can assume that the first of the conditions (2.1) is satisfied if there exist a necessary number of integrals for Eqs. (1.6), (1.7) at $\mu = 0$ [3, 5].

Under condition (1.9) a periodic solution in integral powers of μ does not exist. Let us prove the absence of periodic solution in fractional powers of μ. The proof is carried out by the scheme used in [1, 5]; we indicate the main features. The desired periodic solution of Eqs. (1.6), (1.7) with the initial conditions $v(0) = \alpha$, $z(0) = \beta$, where

β is a sufficiently small constant vector, is written as

$$v(t, \alpha, \beta, \mu) = \varphi(t)\alpha + v_*(t, \alpha, \beta) + \mu \sum_{\nu\geqslant 0} A^{(\nu)}(t, \alpha, \beta) + \mu^2(\cdots) \qquad (2.2)$$

$$z(t, \alpha, \beta, \mu) = f(t)\beta + z_*(t, \alpha, \beta) + \mu(\ldots)$$

Here v_*, z_*, $A^{(\nu)}$ are analytic functions of α, β (the order of v_*, z_* is no less than second, the order of $A^{(\nu)}$ equals ν), $f(t)$ is some $((n+2r)\times(n+2r))$-matrix, depending on t.

We have [1, 5]
$$[\mathring{A}] = 2\pi\delta, \quad z(t, \alpha, \beta, 0) \equiv 0 \qquad Z(v, 0) \equiv 0 \qquad (2.3)$$

From the periodicity conditions $[z] = 0$ we define, with due regard to (2.3), the analytic vector
$$\beta = \beta(\alpha, \mu), \qquad \beta(\alpha, 0) = 0$$

and we substitute into the equation $[v] = 0$ having the form

$$v(2\pi, \alpha, \beta(\alpha, 0), 0) - \alpha + \mu(2\pi\delta + \Phi(\alpha, \mu)) = 0 \qquad (2.4)$$

where Φ is an analytic function of α, μ. Using formula (1.5) we replace the quantity 2π in the following manner:

$$2\pi = p_* T + \Omega, \qquad \Omega = \varkappa - 2\pi\rho_*\lambda^{-1}h(\alpha)$$
$$\varkappa = 2\pi(\lambda - p_*)\lambda^{-1}$$

After an expansion into a series in powers of Ω (with due regard to Eq. (1.6)) condition (2.4) takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n!}(\varkappa B)^n \alpha + \cdots + \mu(2\pi\delta + \Phi(\alpha, \mu)) = 0 \qquad (2.5)$$

where we have not written out terms of higher than first order in α. Using formula (1.8) we compute the coefficient of α, i.e. the absolutely-convergent matrix series

$$\sum_{n=1}^{\infty} \frac{1}{n!}(\varkappa B)^n = \exp(\varkappa B) - 1 = \operatorname{diag}(B_0, \Gamma)$$

$$\Gamma = \operatorname{diag}(\Gamma_1, \ldots, \Gamma_{m-r}), \quad \Gamma_i = \left\| \begin{matrix} \cos\varkappa q_i - 1 & -\sin\varkappa q_i \\ \sin\varkappa q_i & \cos\varkappa q_i - 1 \end{matrix} \right\|$$
$$(i = 1, \ldots, m-r)$$

In notation (c) Eq. (2.5) can finally be written as the system

$$\mu R(\alpha, \mu) \equiv \mu(2\pi\delta^{(1)} + \Phi^{(1)}(\alpha, \mu)) = 0 \qquad (2.6)$$

$$\Gamma\alpha^{(2)} + \cdots + \mu(2\pi\delta^{(2)} + \Phi^{(2)}(\alpha, \mu)) = 0 \qquad (2.7)$$

In Eq. (2.6) terms not depending on μ vanish by virtue of the first condition in (2.1) and of the equality $\beta(\alpha, 0) = 0$).

Under the condition that among the numbers $q_1, \ldots, q_{m-r}$ there are no multiples of the quantity $\lambda(\lambda - p_*)^{-1}$, we have

$$|\Gamma| = 2^{m-r} \prod_{i=1}^{m-r} (1 - \cos\varkappa q_i) \neq 0$$

From Eq. (2.7) we find the analytic vector
$$\alpha^{(2)} = \alpha^{(2)}(\alpha^{(1)}, \mu), \qquad \alpha^{(2)}(0, 0) = 0 \qquad (2.8)$$

and we substitute it into (2.6). We obtain the condition

$$R_*(\alpha^{(1)}, \mu) \equiv 2\pi\delta^{(1)} + \Phi_*^{(1)}(\alpha^{(1)}, \mu) = 0$$

which no vector $\alpha^{(1)}(\mu)$, $\alpha^{(1)}(0) = 0$, can satisfy (under the resonance conditions $\delta^{(1)} \neq 0$). We note that the conclusion obtained is valid for any $\xi \leqslant l - 2r$ in (1.4).

3. If the first condition in (2.1) is not satisfied, then terms independent of μ are present in (2.6). Consequently, the following theorem is valid.

Theorem 3.1. If among the numbers $q_1, \ldots, q_{m-r}$ there are no multiples of the quantity $\lambda(\lambda - p_*)$ and $V^{(1)}(v, 0) \neq 0$, then the number and the form of the desired periodic solutions (in fractional powers of μ) of the system (1.6), (1.7) are determined by the number and the form of the solutions of the new Eq. (2.6) (after the substitution of (2.8)).

Notes. 1. Constraint (2.1) on the vector $\delta^{(1)}$ is not imposed. When $\delta^{(1)} = 0$, only the practical application of Theorem 3 is hampered since it is necessary to compute the first terms in the expansion of the function $\Phi^{(1)}(\alpha, \mu)$ [5].

2. We can ascertain the regularity of the formation of the lower order terms (not depending on μ) in Eq. (2.6) (see, for example, the computation of the coefficient of α in (2.5)). However, the obtaining of these terms in explicit form proves to be inexpedient in view of their cumbersomeness. It is more convenient to examine Eq. (2.6) in application to actual systems. Here we should take into account that in (2.8) the order of the function $\alpha^{(2)}(\alpha^{(1)}, 0)$ is not lower than the order of the function $V^{(2)}(v^{(1)}, 0, 0)$ in (1.6). The latter fact essentially facilitates the composition of Eq. (2.6). For example, suppose that to within higher order terms

$$V^{(2)}(v^{(1)}, 0, 0) = V^{2s}(v^{(1)}) + \ldots, \qquad h(\alpha) = h^{(k)}(\alpha) + \ldots \qquad k \geqslant s$$

where $V^{(2s)}$ are terms of sth order in $v^{(1)}$, $h^{(k)}$ are terms of kth order in α. From Eq. (2.7) we find the vector $\alpha^{(2)}(\alpha^{(1)}, \mu)$

$$\alpha^{(2)}(\alpha^{(1)}, 0) = \alpha^{(2s)}(\alpha^{(1)}) + \ldots$$

If the expansion of the function $V^{(1)}(v^{(1)}, 0, 0)$ starts with terms of order $p \leqslant s$

$$V^{(1)}(v^{(1)}, 0, 0) = V^{(1p)}(v^{(1)}) + \ldots$$

then Eq. (2.6) has the form

$$V^{(1p)}(\alpha^{(1)}) + \ldots + \mu\,(2\pi\delta^{(1)} + \Phi^{(1)}(\alpha^{(1)}, \mu)) = 0$$

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REFERENCES

1. Malkin, I.G., Certain Problems in the Theory of Nonlinear Oscillations. Moscow, Gostekhizdat, 1956.
2. Shimanov, S.N., On the generalization of a proposition due to Liapunov regarding the existence of periodic solutions. PMM Vol. 23, №2, 1959.
3. Liapunov, A.M., General Problem of the Stability of Motion. Moscow-Leningrad, Gostekhizdat, 1950.
4. Starzhinskii, V.M., Liapunov systems with damping. PMM Vol. 36, №2, 1972.
5. Nustrov, V.S., On periodic solutions of systems close to nonlinear systems with principal resonance. Izv. Vuzov, Ser. Matem., №3, 1970.

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